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11. Generalized class formations and higher class field theory

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Let K ($K = K_n, K_{n-1}, \dots, K_0$) be an n-dimensional local field (whose last residue field is finite of characteristic p).

The following theorem can be viewed as a generalization to higher dimensional local fields of the fact $\mathrm{Br}(F) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ for classical local fields F with finite residue field (see section 5).

Theorem (Kato)**.** *There is a canonical isomorphism*

$$h: H^{n+1}(K, \mathbb{Q}/\mathbb{Z}(n)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

Kato established higher local reciprocity map (see section 5 and [K1, Th. 2 of §6] (two-dimensional case), [K2, Th. II], [K3, §4]) using in particular this theorem.

In this section we deduce the reciprocity map for higher local fields from this theorem and Bloch–Kato's theorem of section 4. Our approach which uses generalized class formations simplifies Kato's original argument.

We use the notations of section 5. For a complex $X^\cdot$ the shifted-by-n complex $X^\cdot[n]$ is defined as $(X^\cdot[n])^q = X^{n+q}, d_{X^\cdot[n]} = (-1)^n d_{X^\cdot}$. For a (pro-)finite group G the derived category of G-modules is denoted by $D(G)$.

11.0. Classical class formations

We begin with recalling briefly the classical theory of class formations.

A pair (G, C) consisting of a profinite group G and a discrete G-module C is called a *class formation* if

(C1) $H^1(H, C) = 0$ for every open subgroup H of G.

(C2) There exists an isomorphism $\mathrm{inv}_H: H^2(H, C) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ for every open subgroup H of G.

(C3) For all pairs of open subgroups $V \leqslant U \leqslant G$ the diagram

$$\begin{array}{ccc} H^2(U,C) & \xrightarrow{\text{res}} & H^2(V,C) \\ \downarrow{\scriptstyle \mathrm{inv}_U} & & \downarrow{\scriptstyle \mathrm{inv}_V} \\ \mathbb{Q}/\mathbb{Z} & \xrightarrow{\times|U:V|} & \mathbb{Q}/\mathbb{Z} \end{array}$$

is commutative.

Then for a pair of open subgroups $V \leqslant U \leqslant G$ with V normal in U the group $H^2(U/V,C^V) \simeq \ker(H^2(U,C) \to H^2(V,C))$ is cyclic of order $|U:V|$. It has a canonical generator $u_{L/K}$ which is called the *fundamental class*; it is mapped to $1/|L:K| + \mathbb{Z}$ under the composition

$$H^2(U/V,C^V) \xrightarrow{\text{inf}} H^2(U,C) \xrightarrow{\mathrm{inv}_U} \mathbb{Q}/\mathbb{Z}.$$

Cup product with $u_{L/K}$ induces by the Tate–Nakayama lemma an isomorphism

$$\widehat{H}^{q-2}(U/V,\mathbb{Z}) \xrightarrow{\sim} \widehat{H}^q(U/V,C^V).$$

Hence for $q = 0$ we get $C^U/\,\mathrm{cor}_{U/V}(C^V) \xrightarrow{\sim} (U/V)^{\mathrm{ab}}$.

An example of a class formation is the pair $(G_K, \mathbb{G}_m)$ consisting of the absolute Galois group of a local field K and the G_K-module $\mathbb{G}_m = (K^{\mathrm{sep}})^*$. We get an isomorphism

$$K^*/N_{L/K}L^* \xrightarrow{\sim} \mathrm{Gal}(L/K)^{\mathrm{ab}}$$

for every finite Galois extension L/K.

In order to give an analogous proof of the reciprocity law for higher dimensional local fields one has to work with complexes of modules rather than a single module.

The concepts of the class formations and Tate's cohomology groups as well as the Tate–Nakayama lemma have a straightforward generalization to bounded complexes of modules. Let us begin with Tate's cohomology groups (see [Kn] and [Ko1]).

11.1. Tate's cohomology groups

Let G be a finite group. Recall that there is an exact sequence (called a complete resolution of G)

$$X^{\cdot} \qquad\qquad \ldots \to X^{-2} \to X^{-1} \to X^0 \to X^1 \to \ldots$$

of free finitely generated $\mathbb{Z}[G]$-modules together with a map $X^0 \to \mathbb{Z}$ such that the sequence

$$\cdots \to X^{-1} \to X^0 \to \mathbb{Z} \to 0$$

is exact.

Definition. Let G be a finite group. For a a bounded complex

$$A^{\cdot} \qquad \qquad \ldots \to A^{-1} \to A^0 \to A^1 \to \ldots$$

of G-modules Tate's cohomology groups $\widehat{H}^q(G, A^{\cdot})$ are defined as the (hyper-)cohomology groups of the single complex associated to the double complex

$$Y^{i,j} = \mathrm{Hom}_G(X^{-i}, A^j)$$

with suitably determined sign rule. In other words,

$$\widehat{H}^q(G, A^{\cdot}) = H^q(\mathrm{Tot}(\mathrm{Hom}(X^{\cdot}, A^{\cdot}))^G).$$

Remark. If A is a G-module, then $\widehat{H}^q(G, A^{\cdot})$ coincides with ordinary Tate's cohomology group of G with coefficients in A where

$$A^{\cdot} \qquad \qquad \ldots \to 0 \to A \to 0 \to \ldots \qquad (A \text{ is at degree } 0).$$

Lemma (Tate–Nakayama–Koya, [Ko2])**.** *Suppose that*

(i) $\widehat{H}^1(H, A^{\cdot}) = 0$ *for every subgroup* H *of* G*;*

(ii) *there is* $a \in \widehat{H}^2(G, A^{\cdot})$ *such that* $\mathrm{res}_{G/H}(a)$ *generates* $\widehat{H}^2(H, A^{\cdot})$ *and is of order* $|H|$ *for every subgroups* H *of* G.

Then

$$\widehat{H}^{q-2}(G, \mathbb{Z}) \xrightarrow{\cup a} \widehat{H}^q(G, A^{\cdot})$$

is an isomorphism for all q.

11.2. Generalized notion of class formations

Now let G be a profinite group and $C^{\cdot}$ a bounded complex of G-modules.

Definition. The pair $(G, C^{\cdot})$ is called a *generalized class formation* if it satisfies (C1)–(C3) above (of course, we have to replace cohomology by hypercohomology).

As in the classical case the following lemma yields an abstract form of class field theory

Lemma. *If* $(G, C^{\cdot})$ *is a generalized class formation, then for every open subgroup* H *of* G *there is a canonical map*

$$\rho_H \colon H^0(H, C^{\cdot}) \to H^{\mathrm{ab}}$$

such that the image of ρ_H *is dense in* H^{ab} *and such that for every pair of open subgroups* $V \leqslant U \leqslant G$, V *normal in* U, ρ_U *induces an isomorphism*

$$H^0(U, C^{\cdot}) / \mathrm{cor}_{U/V}\ H^0(V, C^{\cdot}) \xrightarrow{\sim} (U/V)^{\mathrm{ab}}.$$

11.3. Important complexes

In order to apply these concepts to higher dimensional class field theory we need complexes which are linked to K-theory as well as to the Galois cohomology groups $H^{n+1}(K,\,\mathbb{Q}/\mathbb{Z}(n))$. Natural candidates are the Beilinson–Lichtenbaum complexes.

Conjecture ([Li1]). *Let K be a field. There is a sequence of bounded complexes $\mathbb{Z}(n)$, $n \geqslant 0$, of G_K-modules such that*

(a) $\mathbb{Z}(0) = \mathbb{Z}$ *concentrated in degree* 0*;* $\mathbb{Z}(1) = \mathbb{G}_m[-1]$;
(b) $\mathbb{Z}(n)$ *is acyclic outside* $[1,n]$;
(c) *there are canonical maps* $\mathbb{Z}(m) \otimes^{\mathbb{L}} \mathbb{Z}(n) \to \mathbb{Z}(m+n)$;
(d) $H^{n+1}(K,\mathbb{Z}(n)) = 0$;
(e) *for every integer m there is a triangle* $\mathbb{Z}(n) \xrightarrow{m} \mathbb{Z}(n) \longrightarrow \mathbb{Z}/m(n) \longrightarrow \mathbb{Z}(n)[1]$ *in* $D(G_K)$;
(f) $H^n(K,\mathbb{Z}(n))$ *is identified with the Milnor K-group $K_n(K)$.*

Remarks. 1. This conjecture is very strong. For example, (d), (e), and (f) would imply the Milnor–Bloch–Kato conjecture stated in 4.1.

2. There are several candidates for $\mathbb{Z}(n)$, but only in the case where $n = 2$ proofs have been given so far, i.e. there exists a complex $\mathbb{Z}(2)$ satisfying (b), (d), (e) and (f) (see [Li2]).

By using the complex $\mathbb{Z}(2)$ defined by Lichtenbaum, Koya proved that for 2-dimensional local field K the pair $(G_K,\mathbb{Z}(2))$ is a class formation and deduced the reciprocity map for K (see [Ko1]). Once the existence of the $\mathbb{Z}(n)$ with the properties (b), (d), (e) and (f) above is established, his proof would work for arbitrary higher dimensional local fields as well (i.e. $(G_K,\mathbb{Z}(n))$ would be a class formation for an n-dimensional local field K).

However, for the purpose of applications to local class field theory it is enough to work with the following simple complexes which was first considered by B. Kahn [Kn].

Definition. Let $\check{\mathbb{Z}}(n) \in D(G_K)$ be the complex $\mathbb{G}_m^{\overset{\mathbb{L}}{\otimes} n}[-n]$.

Properties of $\check{\mathbb{Z}}(n)$.

(a) it is acyclic outside $[1,n]$;
(b) for every m prime to the characteristic of K if the latter is non-zero, there is a triangle

$$\check{\mathbb{Z}}(n) \xrightarrow{m} \check{\mathbb{Z}}(n) \to \mathbb{Z}/m(n) \to \check{\mathbb{Z}}(n)[1]$$

in $D(G_K)$;

(c) for every m as in (b) there is a commutative diagram

$$\begin{array}{ccc} K^{*\otimes n} & \longrightarrow & H^n(K, \check{\mathbb{Z}}(n)) \\ {\scriptstyle \mathrm{pr}}\big\downarrow & & \big\downarrow \\ K_n(K)/m & \longrightarrow & H^n(K, \mathbb{Z}/m(n)). \end{array}$$

where the bottom horizontal arrow is the Galois symbol and the left vertical arrow is given by $x_1 \otimes \cdots \otimes x_n \mapsto \{x_1, \ldots, x_n\} \mod m$.

The first two statements are proved in [Kn], the third in [Sp].

11.4. Applications to n-dimensional local class field theory

Let K be an n-dimensional local field. For simplicity we assume that $\mathrm{char}(K) = 0$. According to sections 3 and 5 for every finite extension L of K there are isomorphisms

(1) $$K_n(L)/m \xrightarrow{\sim} H^n(L, \mathbb{Z}/m(n)), \quad H^{n+1}(L, \mathbb{Q}/\mathbb{Z}(n)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

Lemma. $(G, \check{\mathbb{Z}}(n)[n])$ *is a generalized class formation.*

The triangle (b) above yields short exact sequences

$$0 \to H^i(K, \check{\mathbb{Z}}(n))/m \to H^i(K, \mathbb{Z}/m(n)) \to {}_mH^{i+1}(K, \check{\mathbb{Z}}(n)) \to 0$$

for every integer i. (1) and the diagram (c) show that ${}_mH^{n+1}(K, \check{\mathbb{Z}}(n)) = 0$ for all $m \neq 0$. By property (a) above $H^{n+1}(K, \check{\mathbb{Z}}(n))$ is a torsion group, hence $= 0$. Therefore (C1) holds for $(G, \check{\mathbb{Z}}(n)[n])$. For (C2) note that the above exact sequence for $i = n+1$ yields $H^{n+1}(K, \mathbb{Z}/m(n)) \to {}_mH^{n+2}(K, \check{\mathbb{Z}}(n))$. By taking the direct limit over all m and using (1) we obtain

$$H^{n+2}(K, \check{\mathbb{Z}}(n)) \xrightarrow{\sim} H^{n+1}(K, \mathbb{Q}/\mathbb{Z}(n)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

Now we can establish the reciprocity map for K: put $C^{\cdot} = \check{\mathbb{Z}}(n)[n]$ and let L/K be a finite Galois extension of degree m. By applying abstract class field theory (see the lemma of 11.2) to $(G, C^{\cdot})$ we get

$$\begin{aligned} K_n(K)/N_{L/K}K_n(L) &\xrightarrow{\sim} H^n(K, \mathbb{Z}/m(n))/\operatorname{cor} H^n(L, \mathbb{Z}/m(n)) \\ &\xrightarrow{\sim} H^0(K, C^{\cdot})/m/\operatorname{cor} H^0(L, C^{\cdot})/m \xrightarrow{\sim} \mathrm{Gal}(L/K)^{\mathrm{ab}}. \end{aligned}$$

For the existence theorem see the previous section or Kato's paper in this volume.

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